DESIGN OF BEAMS

Design formulae for beams

Notation

L	=	length of span in millimetres
W	=	total distributed or point load in Newtons
W_1 or W_2	=	point load in Newtons
Σ	=	resultant of point loads in Newtons
R_A, R_B, R_C, etc.	=	reaction at A, B, or C etc, in Newtons
F	=	shearing force in Newtons
m	=	applied moment in Newton millimetres
M_x	=	bending moment in Newton millimetres *at distance X from the left hand support A*
δ_x	=	deflection in millimetres
i_x	=	slope in radians
M_A or M_B	=	end fixing moment in Newton millimetres
M_{max}	=	maximum bending moment in Newton millimetres
$M_{max\ max}$	=	absolute maximum bending moment in Newton millimetres
M_{load}	=	bending moment under the load in Newton millimetres
δ_{max}	=	maximum deflection in millimetres
$\delta_{max\ max}$	=	absolute maximum deflection in millimetres
$\delta_{negative}$	=	negative (i.e. upward) deflection in millimetres
i_A or i_B	=	slope at A or at B in radians
E	=	modulus of elasticity, 2.1×10^5 N/mm^2
I	=	constant moment of inertia of uniform section beam in mm^4

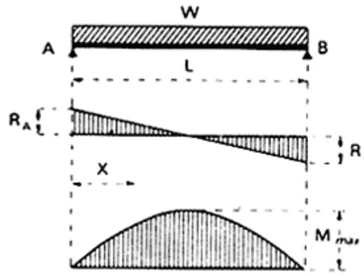
Sign convention

Loads	+	positive when acting downward
Support Reaction	+	positive when acting upward
Shearing Force	+	positive on a section where the upward left hand support reaction is greater than the algebraic sum of external loads located left of that section
Bending Moment	+	positive (shown above base line on diagrams) when causing convexity downward
Deflection	+	positive when downward
Slope		appropriate values in radians are given, but the signs depend upon which support or which section is being considered, and can be readily ascertained by inspection

Where space permits, general equations for M_x and i_x at any point of the beam, and also the equation to the elastic line (δ_x), have been included.

Values for Slope. These may be used in evaluating the angle of rotation for rubber bearings and similar constructional elements.

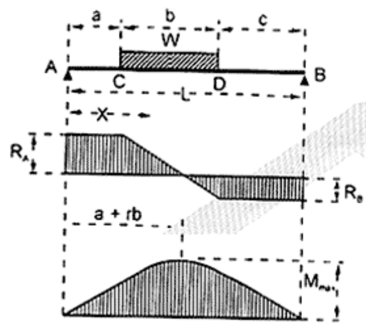
Simply supported beam



Uniform load on full span

Span	=	L
Total uniform load	=	W
$R_A = R_B$	=	$\frac{W}{2}$
at mid-span: M_{max}	=	$\frac{WL}{8}$
at mid-span: δ_{max}	=	$\frac{5}{384} \cdot \frac{WL^3}{EI}$
$i_A = i_B$	=	$\frac{WL^2}{24EI}$
at X from A: M_x	=	$\frac{WX}{2L}(L-X)$
at X from A: δ_x	=	$\frac{WX}{24EIL}(X^3-2X^2L+L^3)$
at X from A: i_x	=	$\frac{W}{24EIL}(4X^3-6X^2L+L^3)$

Simply supported beam



Uniform load on part of span

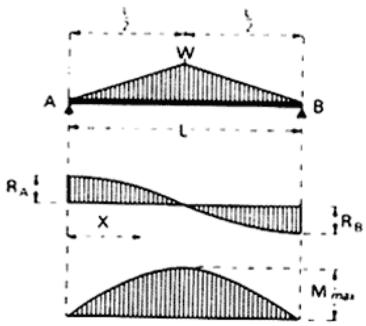
Span	=	L
Total uniform load	=	W
Let r	=	$\frac{0.5b+c}{L}$
$R_A = Wr$; R_B	=	$W(1-r)$
at $X = a + rb$, M_{max}	=	$Wr(a+0.5rb)$

$i_A = \frac{Wr}{6EI}(L^2-c^2-Lbr)$; $i_B = \frac{W(1-r)}{6EI}[L^2-a^2-Lb(1-r)]$

Equation to elastic line between C and D, i.e. $a \leq X \leq a+b$

$$\delta x = \frac{W}{24EIb}\left[X^4-4(a+rb)X^3+6a^2X^2+4\left\{rb(L^2-c^2-cb-\frac{b^2}{2})-a^3\right\}X+a^4\right]$$

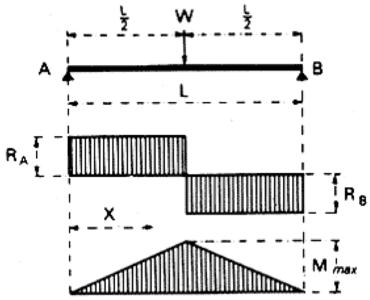
Simply supported beam



Triangular load on full span

Span	=	L
Total load	=	W
$R_A = R_B$	=	$\frac{W}{2}$
at mid-span: M_{max}	=	$\frac{WL}{6}$
at mid-span: δ_{max}	=	$\frac{WL^3}{60EI}$
$i_A = i_B$	=	$\frac{5WL^2}{96EI}$
at X from A between A and centre: M_x	=	$\frac{WX}{6L^2}(3L^2-4X^2)$
at X from A between A and centre: δ_x	=	$\frac{WX}{480EIL^2}(16X^4-40X^2L^2+25L^4)$
at X from A between A and centre: i_x	=	$\frac{W}{96EIL^2}(16X^4-24X^2L^2+5L^4)$

Simply supported beam — Point load at mid-span



Span $= L$

Point load $= W$

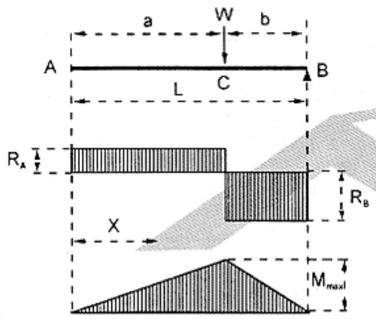
$R_A = R_B = \dfrac{W}{2}$

at mid-span: $M_{max} = \dfrac{WL}{4}$; $\delta_{max} = \dfrac{1}{48} \cdot \dfrac{WL^3}{EI}$

$i_A = i_B = \dfrac{WL^2}{16EI}$

at X from A between A and centre: $M_x = \dfrac{WX}{2}$; $\delta_x = \dfrac{WX}{48EI}(3L^2-4X^2)$; $i_x = \dfrac{W}{16EI}(L^2-4X^2)$

Simply supported beam — Point load at any position



Span $= L$

Point load $= W$

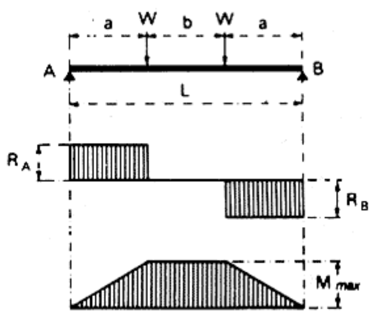
$R_A = \dfrac{Wb}{L}$ $\quad R_B = \dfrac{Wa}{L}$

at C under load: $M_{max} = \dfrac{Wab}{L}$; $\delta_c = \dfrac{Wa^2b^2}{3EIL}$

$i_A = \dfrac{Wab}{6EIL}(L+b)$; $\quad i_B = \dfrac{Wab}{6EIL}(L+a)$

When a>b, δ_{max} is at X from A: $\delta_{max} = \dfrac{Wab(L+b)}{27EIL}\sqrt{3a(L+b)}$; $X = \sqrt{\dfrac{a(L+b)}{3}}$

Simply supported beam — Two equal symmetrical point loads



Span $= L$

Two point loads, each $= W$

$R_A = R_B = W$

M_{max} over length b $= Wa$

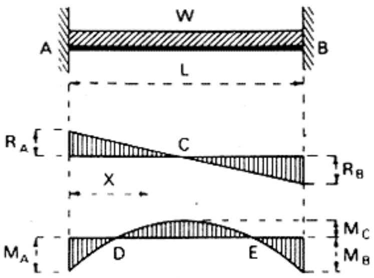
δ_{max} at mid-span $= \dfrac{Wa}{24EI}(3L^2-4a^2)$

δ under either load $= \dfrac{Wa^2}{6EI}(3L-4a)$

$i_A = i_B = \dfrac{Wa}{2EI}(L-a)$

If $a=b=\dfrac{L}{3}$, $\delta_{max} = \dfrac{23}{648} \cdot \dfrac{WL^3}{EI}$

Beam fixed at both ends — Uniform load on full span



Span = L

Total uniform load = W

$R_A = R_B = \frac{W}{2}$

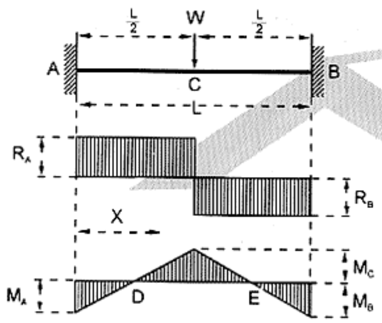
$M_A = M_B = -\frac{WL}{12}$

at mid-span:
- $M_c = \frac{WL}{24}$
- $\delta_{max} = \frac{WL^3}{384EI}$

at X from A:
- $M_x = \frac{W}{12L}(L^2-6LX+6X^2)$
- $\delta_x = \frac{WX^2}{24EIL}(L-X)^2$
- $i_x = \frac{WX}{12EIL}(L^2-3LX+2X^2)$

at 0.211L from either end $M_D=M_E=0$

Beam fixed at both ends — Point load at mid-span



Span = L

Point load = W

$R_A = R_B = \frac{W}{2}$

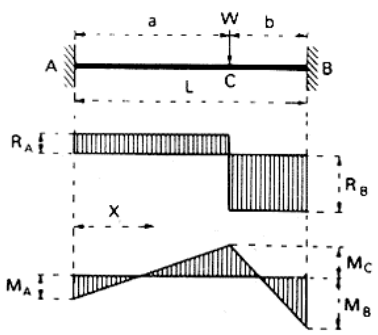
$M_A = M_B = -\frac{WL}{8}$

at mid-span:
- $M_c = \frac{WL}{8}$
- $\delta_{max} = \frac{WL^3}{192EI}$

at X from A between A and C:
- $M_x = \frac{W}{8}(4X-L)$
- $\delta_x = \frac{WX^2}{48EI}(3L-4X)$
- $i_x = \frac{WX}{8EI}(L-2X)$

at 0.25L from either end $M_D=M_E=0$

Beam fixed at both ends — Point load at any position



Span = L

Point load = W

$R_A = \frac{Wb^2(L+2a)}{L^3}$ $R_B = \frac{Wa^2(L+2b)}{L^3}$

$M_A = -\frac{Wab^2}{L^2}$ $M_B = -\frac{Wa^2b}{L^2}$

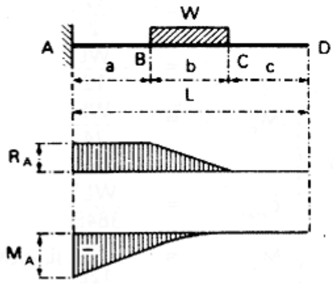
at C, under load, $M_C = \frac{2Wa^2b^2}{L^3}$

at X from A between A and C:
- $M_x = -\frac{Wab^2}{L^2} + \frac{Wb^2(L+2a)X}{L^3}$
- $\delta_x = \frac{Wb^2X^2[3La-(L+2a)X]}{6EIL^3}$
- $i_x = \frac{Wb^2X[2La-(L+2a)X]}{2EIL^3}$

$\delta_{max} = \frac{2Wa^3b^2}{3EI(L+2a)^2}$

when a>b the maximum deflection is at $X=\frac{2La}{L+2a}$

Cantilever — Uniform load on part of span



Uniform load on part of span

Span = L

Uniform load = W

R_A = W

$M_A = -W\left(a+\frac{b}{2}\right)$

$i_C = i_D = \frac{W}{6EI}(3a^2+3ab+b^2)$

$\delta_D = \frac{W}{24EI}[8a^3+18a^2b+12ab^2+3b^3+4c(3a^2+3ab+b^2)]$

Special case: Uniform load on full span

Span = L = b

a = c = 0

Uniform load = W

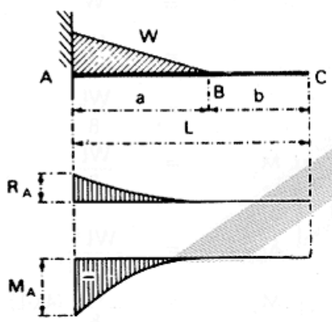
R_A = W

$M_A = -\frac{WL}{2}$

$i_D = \frac{WL^2}{6EI}$

$\delta_D = \frac{WL^3}{8EI}$

Cantilever — Triangular load on part of span



Triangular load on part of span

Span = L

Triangular load = W

R_A = W

$M_A = -\frac{Wa}{3}$

$\delta_C = \frac{Wa^2}{15EI}\left(L+\frac{b}{4}\right)$

$i_B = i_C = \frac{Wa^2}{12EI}$

Special case: Triangular load on full span

Span = L = a

b = 0

Triangular load = W

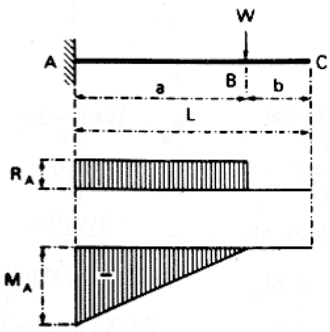
R_A = W

$M_A = -\frac{WL}{3}$

$\delta_C = \frac{WL^3}{15EI}$

$i_C = \frac{WL^2}{12EI}$

Cantilever — Point load at any position



Point load at any position

Span = L

Point load = W

R_A = W

M_A = - Wa

$\delta_C = \frac{Wa^2}{3EI}\left(L+\frac{b}{2}\right)$

$i_B = i_C = \frac{Wa^2}{2EI}$

Special case: Point load at free end

Span = L = a

b = 0

Point load = W

R_A = W

M_A = -WL

$\delta_C = \frac{WL^3}{3EI}$

$i_C = \frac{WL^2}{2EI}$

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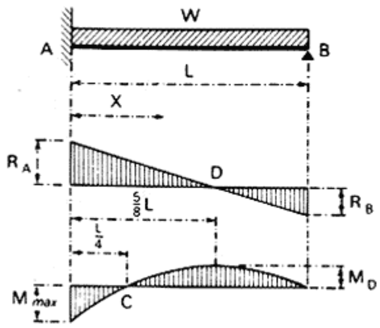
Propped cantilever — **Uniform load on full span**



Span $= L$

Total uniform load $= W$

$R_A = \frac{5}{8}W \qquad R_B = \frac{3}{8}W$

at A $\qquad M_{max} = -\frac{WL}{8}$

at $\frac{5}{8}L$ from A $\qquad M_D = \frac{9}{128}WL$

at 0.5785L from A $\qquad \delta_{max} = \frac{WL^3}{185EI}$

at B $\qquad i_B = \frac{WL^2}{48EI}$

at X from A:

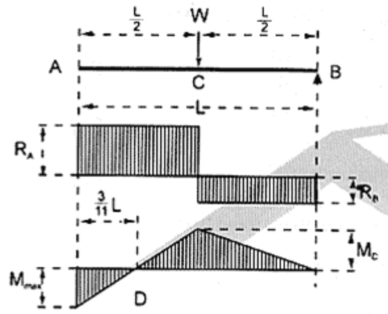
$M_x = \frac{W}{8L}(L^2-5LX+4X^2)$

$\delta_x = \frac{WX^2}{48EIL}(3L^2-5LX+2X^2)$

$i_x = \frac{WX}{48EIL}(6L^2-15LX+8X^2)$

at $\frac{1}{4}L$ from A, $M_C = 0$

Propped cantilever — **Point load at mid-span**



Span $= L$

Point load $= W$

$R_A = \frac{11}{16}W \qquad R_B = \frac{5}{16}W$

at A $\qquad M_{max} = -\frac{3}{16}WL$

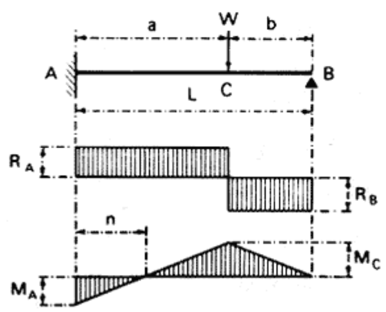
at mid-span under load: $M_C = \frac{5}{32}WL$, $\delta_C = \frac{7WL^3}{768EI}$

at 0.5528L from A $\qquad \delta_{max} = \frac{WL^3}{107EI}$

at B $\qquad i_B = \frac{WL^2}{32EI}$

at $\frac{3}{11}L$ from A, $M = 0$

Propped cantilever — **Point load at any position**



Span $= L$

Point load $= W$

$R_A = \frac{Wb(3L^2-b^2)}{2L^3} \qquad R_B = \frac{Wa^2(2L+b)}{2L^3}$

$M_A = -\frac{Wab(L+b)}{2L^2} \qquad M_C = \frac{Wa^2b(2L+b)}{2L^3}$

$i_B = \frac{Wa^2b}{4EIL}$

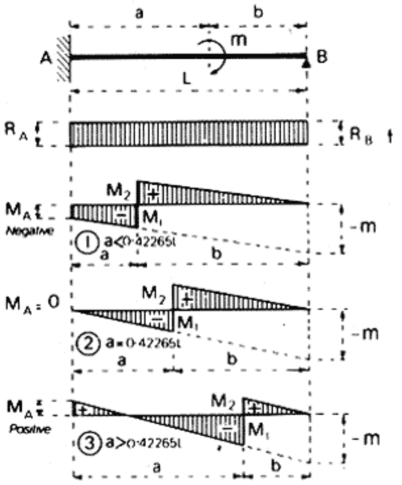
Absolute max deflection is under the load when $a=b\sqrt{2}=0.5858L$: $\delta_{max\ max} = \frac{WL^3}{102EI}$

When $a>b\sqrt{2}$ max deflection is between A and C: $\delta_{max} = \frac{Wa^3b}{3EI} \cdot \frac{(L+b)^3}{(3L^2-b^2)^2}$

When $a<b\sqrt{2}$ max deflection is between C and B: $\delta_{max} = \frac{Wa^2b}{6EI}\sqrt{\frac{b}{2L+b}}$

at $n = aL\frac{L+b}{3L^2-b^2}$ from A, $M = 0$

Propped cantilever — Moment applied at any point



Span = L

Applied moment = m

$M_A = \frac{L^2-3b^2}{2L^2} m$ $\qquad i_B = \frac{ma}{4EIL}(2b-a)$

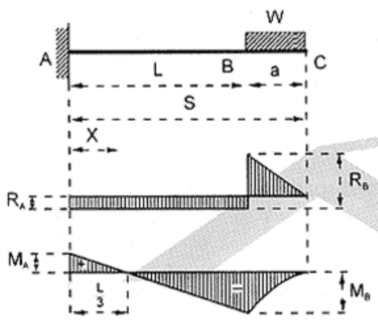
$R_A = -R_B = -\frac{3(L^2-b^2)}{2L^3} m = -\frac{m+M_A}{L}$

$$\begin{cases} M_1 = -\frac{m}{L^3}\left(a^3+\frac{3}{2}a^2b+b^3\right) \\ M_2 = \frac{3mab}{L^3}\left(b+\frac{a}{2}\right) = m+M_1 \end{cases}$$

$$\begin{cases} M_1 = -0.42265m \\ M_2 = 0.57735m \end{cases}$$

$$\left.\begin{matrix} M_1 = \\ M_2 = \end{matrix}\right\} \text{ as for ①}$$

Propped cantilever — Uniform load on length beyond prop



Span = L $\qquad$ Full length = S

Uniform load = W

$R_A = -\frac{3Wa}{4L}$ $\qquad R_B = \frac{W}{L}\left(S-\frac{a}{4}\right)$

$M_A = \frac{Wa}{4}$ $\qquad M_B = -\frac{Wa}{2}$

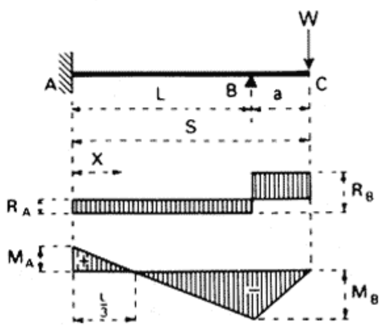
Deflection at C = $\delta_{max} = \frac{Wa^2S}{8EI}$

Max. negative deflection at $X = \frac{2}{3}L$ $\quad \delta_{neg} = -\frac{WL^2a}{54EI}$

Slope at C = $i_C = \frac{Wa}{8EI}\left(S+\frac{a}{3}\right)$

at $X = \frac{L}{3}$ from A, $M = 0$

Propped cantilever — Point load at free end



Span = L $\qquad$ Full length = S

Point load = W

$R_A = -\frac{3Wa}{2L}$ $\qquad R_B = \frac{W}{L}\left(S+\frac{a}{2}\right)$

$M_A = \frac{Wa}{2}$ $\qquad M_B = -Wa$

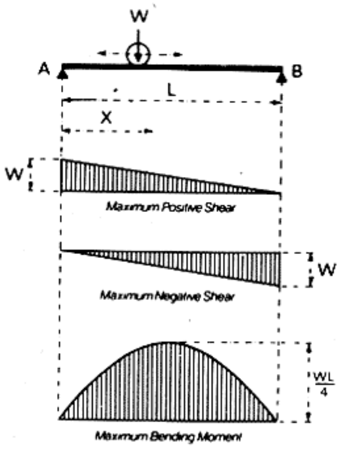
Deflection at C = $\delta_{max} = \frac{Wa^2}{4EI}\left(S+\frac{a}{3}\right)$

Max. negative deflection at $X = \frac{2}{3}L$ $\quad \delta_{neg} = -\frac{WL^2a}{27EI}$

Slope at C = $i_C = \frac{Wa}{4EI}(S+a)$

at $X = \frac{L}{3}$ from A, $M = 0$

Simply supported beam



Single concentrated moving load

Maximum Positive Shear at any section occurs when the load is immediately to the right of the section. Similarly, Maximum Negative Shear occurs when the load is to the left. For a section distance X from A:

Positive $F_{X\,max} = W\frac{L-X}{L}$ Negative $F_{X\,max} = -W\frac{X}{L}$

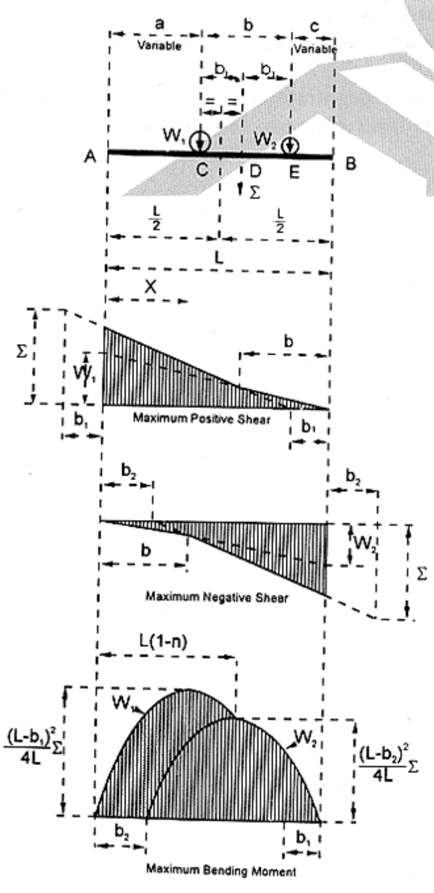
Maximum Bending Moment at any section occurs when the load is over the section. For a section distance X from A:

$$M_{X\,max} = W\frac{X(L-X)}{L}$$

The Absolute Maximum Bending Moment and Deflection occur under the load at mid-span: $M_{max\,max} = \frac{WL}{4}$, $\delta_{max\,max} = \frac{WL^3}{48EI}$

Maximum end slope at A occurs with the load at X=0.42265L from A: $i_{A\,max} = 0.06415\frac{WL^2}{EI}$

Simply supported beam



Two concentrated moving loads

Assume: $W_1 > W_2$ $W_1+W_2=\Sigma$ $W_2 = n\Sigma$
Fixed Distance $b = mL$

$$b_1 = \frac{W_2}{W_1+W_2}b = nmL \qquad b_2 = (m-nm)L$$

Maximum Reaction at A and Absolute Maximum Positive Shear occur when W_1 is immediately to the right of A:

$$R_{A\,max} = \text{Positive } F_{max\,max} = W_1+W_2\frac{L-b}{L}$$

For a section distance X from A:

$X \le L-b$: Positive $F_{max} = \frac{L-X}{L}\Sigma - mW_2$

$L-b \le X$: Positive $F_{max} = \frac{L-X}{L}W_1$

Note:
1. For $R_{B\,max}$, interchange values of W_1 and W_2 in the formula for $R_{A\,max}$
2. For Negative Shear, interchange W_1 and W_2 in formulae for Positive Shear, measuring X from B towards A
3. If $m > \frac{n}{1-n}$ calculate $R_{B\,max}$ and Negative Shear values for W_1 only as single load.

Absolute Maximum Bending Moment occurs under W_1 when that load and the resultant of both loads are equidistant from mid-span (see loading diagram):

$$M_{max\,max} = \frac{(L-b_1)^2}{4L}\Sigma$$

If m<n, the Maximum Bending Moment at any section occurs under one of the loads. For a section distance X from A:

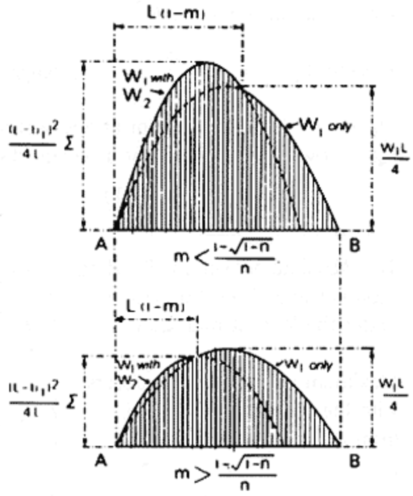
$X \le L(l-n)$

$$M_{max} \text{ under } W_1 = \frac{(L-b_1-X)X}{L}\Sigma$$

$L(l-n) \le X$

$$M_{max} \text{ under } W_2 = \frac{(X-b_2)(L-X)}{L}\Sigma$$

Simply supported beam



Two concentrated moving loads *(continued)*

If m>n, the Maximum Bending Moment at any section always occurs under W_1 (the heavier load), whether W_2 is on or off the span.

For a section distance X from A:

$X \leq L\,(l - m)$: $L\,(l - m) \leq X$

$M_{max} = \frac{(L - b1 - X)X}{L}\,\Sigma$: $M_{max} = \frac{(L - X)X}{L}\,W_1$

If $n<m<\frac{l - \sqrt{l - n}}{n}$ the Absolute Maximum Bending Moment occurs under W1 with W2 on the span.

If $n<m>\frac{l - \sqrt{l - n}}{n}$ the Absolute Maximum Bending Moment occurs under W1 at mid-span with W2 off the span.

Note: When the two loads are equal (W1=W2 and n=½) the critical value of $\frac{l - \sqrt{l - n}}{n} = 0.5858$.

Simply supported beams carrying several moving concentrated loads

The Maximum Reaction and the Maximum Shear due to several moving concentrated loads occur at one support with one of the loads at that support. The location producing the Absolute Maximum must be found by trial.

The Maximum Bending Moment due to several moving concentrated loads occurs under one of the loads when that load and the gravity centre of all loads are equidistant from mid-span. The Absolute Maximum must be determined by trial.

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